Lower Bounds and Separations for Torus Polynomials

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Conjecture (Barrington '89)

MAJORITY $\notin$ ACC^0.

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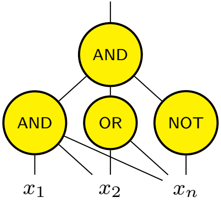

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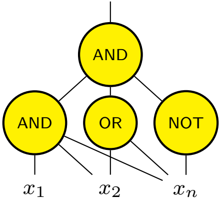

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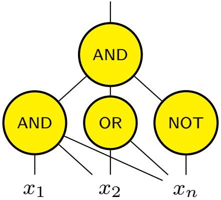

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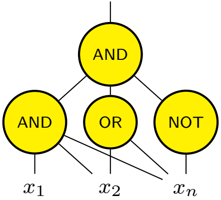

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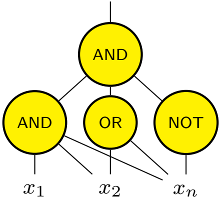

Definition (ACC^0)

- Polynomial size.
- Constant depth.
- Containing AND, OR, NOT and MOD$_m$ gates.

Previous Progress

P - - - - - - - - - - - - - - - - - $\mathsf{AC}^0[p]$ (allow MOD_p)

MOD_6 (Smolensky '87)

MAJORITY (Razborov '87)

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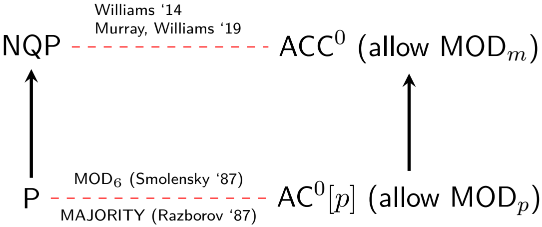

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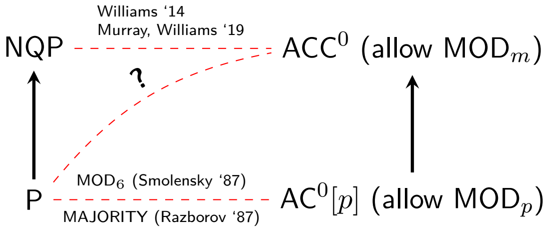

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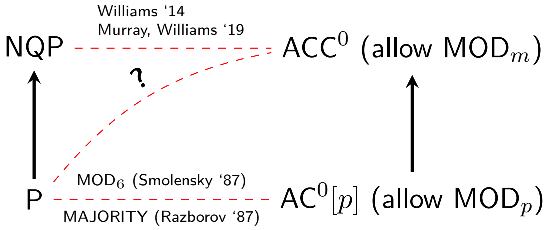

These techniques seem insufficient for MAJORITY $\notin$ ACC^0.

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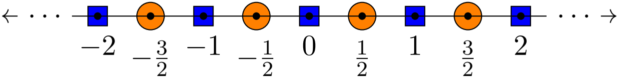

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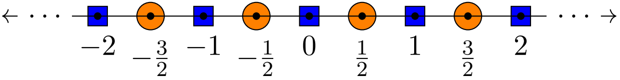

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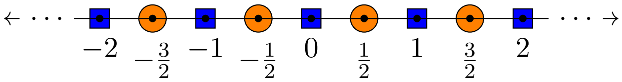

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Theorem (BHLR '19)

All functions in ACC^0 *have polylog-degree torus approximations with inverse-polynomial error.*

Why Torus Polynomials

Theorem (BHLR '19)

For $\varepsilon = \frac{1}{20n}$, *any symmetric torus polynomial* ε*-approximating* MAJORITY *must have degree* $\widetilde{\Omega}\left(\sqrt{n}\right)$.

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Conjecture (BHLR '19)

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- Lower bound iff programs are infeasible iff duals are feasible.

The Family of Duals

- For each Z, find $\gamma \in nullspace(M(n,d))$, such that :

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- Extends the *method of dual polynomials* to torus polynomials.
- Allows for incremental progress.

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- New nullspace vectors supported on a single Hamming layer.
  - Asymmetric construction, unlike previously known.

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  - Strengthens corresponding result from [BHLR ‘19].

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  - Use lattice theory.

Thank you

Questions?