Upper Bound for Torus Polynomials

Vaibhav Krishan

Computer Science and Engineering, IIT Bombay

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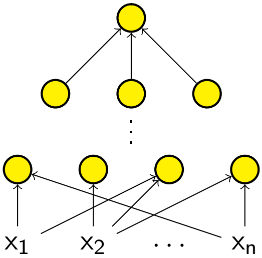
Outline

1. Introduction
2. Our Results
3. Proof
4. Conclusion

Boolean Circuits



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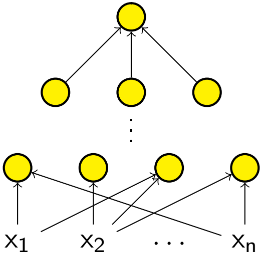

size: # of gates/wires

depth: length of longest path

$\mathcal{G}$: allowed gates

Early Lower Bounds

Definition (AC^0)

$\mathcal{G} = \{\wedge, \vee, \neg\}$.

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Theorem ([FSS84, Ajt83, Yao85, Hås87])

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Theorem ([Raz87, Smo87])

$MAJ \notin AC^0[p]$.

ACC Lower Bounds

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Theorem ([Wil14])

$\mathrm{NEXP} \not\subset \mathrm{ACC}$.

Torus Polynomials

Definition (Torus Polynomial)

$P(x_1, \ldots, x_n) \in \mathbb{R}[X_1, \ldots, X_n]$ is a torus polynomial that ε-approximates f if

$$P(x) - f(x)/2 \in [N(x) - \varepsilon, N(x) + \varepsilon], N(x) \in \mathbb{Z}$$

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Theorem ([BHLR18])

Let $f \in \mathrm{ACC}$ and $\varepsilon = n^{-O(1)}$. Then $deg_\varepsilon(f) \leq (\log(n))^{O(1)}$.

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MidBit$^+$

Definition (MidBit)

$\text{MidBit} : \{0,1\}^n \to \{0,1\}, \ell = \lfloor \log_2(n) \rfloor + 1$

$bin\left(\sum_{i=1}^{n} x_i\right) = b_{\ell-1} \dots b_{\lfloor \ell/2 \rfloor+1} \text{MidBit}(x) b_{\lfloor \ell/2 \rfloor-1} \dots b_0$

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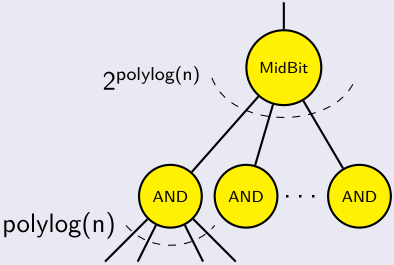
Definition (MidBit$^+$)



Power of MidBit^+

Theorem (GKT'92)

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Lemma

$\mathrm{MAJ} \in \mathrm{MidBit}^+$

Upper Bound for Torus Polynomials

Theorem

Let $\varepsilon < 1/8$. $deg_{\varepsilon}(f) \leq (\log(n))^{O(1)} \implies f \in \mathrm{MidBit}^{+}$

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Theorem

Let $\varepsilon < 1/8$. *Let* $deg_{\varepsilon}(f) = d$. *Then, there is* MidBit^{+} *circuit computing* f*, of the following form:*

- *fan-in of each* AND *gate is bounded by* d*,*
- *fan-in of the* MidBit *gate is bounded by* 2^{2k-1} *where* $2^k = (d+1)n^d/\varepsilon$*,*
- *for* $x \in \{0,1\}^n$*, let* $A(x)$ *be the number of* AND *gates that output 1. Then* $A(x) \equiv f(x)2^{k-1} + E(x) \mod 2^k$*, where* $E(x) \leq 4\varepsilon 2^k$*.*

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Proof

- Let $P = \sum_{\alpha} c_{\alpha} X^{\alpha}$ ε-approximate f.
  $P(x) \in [N(x) + f(x)/2 - \varepsilon, N(x) + f(x)/2 + \varepsilon]$

Proof

- Let $P = \sum_\alpha c_\alpha X^\alpha$ ε-approximate f.
  $P(x) \in [N(x) + f(x)/2 - \varepsilon, N(x) + f(x)/2 + \varepsilon]$
- Take $P_{pos} = P + \varepsilon$.
  $P_{pos}(x) \in [N(x) + f(x)/2, N(x) + f(x)/2 + 2 * \varepsilon]$

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- $bin(P_{pos}(x)) = \ldots 101.f(x)0110010\ldots$

Proof - Continued

- Choose a k. For each c_α, calculate $q_\alpha \in \mathbb{Z}$ such that $\left| c_\alpha - \frac{q_\alpha}{2^k} \right| \leq \frac{1}{2^k}$

Proof - Continued

- Choose a k. For each c_α, calculate $q_\alpha \in \mathbb{Z}$ such that $\left| c_\alpha - \frac{q_\alpha}{2^k} \right| \leq \frac{1}{2^k}$
- $P_{disc}(x) = \sum_\alpha q_\alpha / 2^k X^\alpha$.
  $bin(P_{disc}(x)) = \ldots 101.f(x)0110010\ldots$

Proof - Continued

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- $P_{disc}(x) = \sum_\alpha q_\alpha / 2^k X^\alpha$.
  $bin(P_{disc}(x)) = \ldots 101.f(x)0110010\ldots$
- $P_{int}(x) = 2^k P_{disc}(x) = \sum_\alpha q_\alpha X^\alpha$
  $bin(P_{int}(x)) = \ldots 101f(x)0110$

Proof - Continued

- Choose a k. For each c_α, calculate $q_\alpha \in \mathbb{Z}$ such that $\left|c_\alpha - \frac{q_\alpha}{2^k}\right| \leq \frac{1}{2^k}$
- $P_{disc}(x) = \sum_\alpha q_\alpha/2^k X^\alpha$.
  $bin(P_{disc}(x)) = \ldots 101.f(x)0110010\ldots$
- $P_{int}(x) = 2^k P_{disc}(x) = \sum_\alpha q_\alpha X^\alpha$
  $bin(P_{int}(x)) = \ldots 101f(x)0110$
- Convert to MidBit$^+$.

Parity over Torus Polynomials

Introduction

Our Results

Proof

Conclusion

Theorem

Let $deg_{\varepsilon_1}(f_1) = d_1$ *and* $deg_{\varepsilon_2}(f_2) = d_2$. *Then* $deg_{\varepsilon_1+\varepsilon_2}(f_1 \oplus f_2) \leq \max(d_1, d_2)$

Parity over Torus Polynomials

Theorem

Let $deg_{\varepsilon_1}(f_1) = d_1$ *and* $deg_{\varepsilon_2}(f_2) = d_2$. *Then* $deg_{\varepsilon_1+\varepsilon_2}(f_1 \oplus f_2) \leq \max(d_1, d_2)$

Proof.

Let P_1 ε_1-approximate f_1 and P_2 ε_2-approximate f_2. Consider $P_1 + P_2$. $\square$

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Conclusion

- Power of torus polynomials is upper bounded by MidBit^+.

Conclusion

- Power of torus polynomials is upper bounded by MidBit^{+}.
- Torus polynomials are closed under parity.

Thank You

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References I

Miklós Ajtai, Σ_1^1-*formulae on finite structures*, Annals of pure and applied logic **24** (1983), no. 1, 1–48.

Abhishek Bhrushundi, Kaave Hosseini, Shachar Lovett, and Sankeerth Rao, *Torus polynomials: An algebraic approach to* ACC *lower bounds*, 10th Innovations in Theoretical Computer Science Conference (ITCS 2019), Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

Merrick Furst, James B Saxe, and Michael Sipser, *Parity, circuits, and the polynomial-time hierarchy*, Mathematical systems theory **17** (1984), no. 1, 13–27.

Johan Håstad, *Computational limitations of small-depth circuits*, MIT press, 1987.

References II

Alexander A Razborov, *Lower bounds on the size of bounded depth circuits over a complete basis with logical addition*, Mathematical Notes of the Academy of Sciences of the USSR **41** (1987), no. 4, 333–338.

Roman Smolensky, *Algebraic methods in the theory of lower bounds for boolean circuit complexity*, Proceedings of the nineteenth annual ACM symposium on Theory of computing, 1987, pp. 77–82.

Ryan Williams, *Nonuniform* ACC *circuit lower bounds*, Journal of the ACM (JACM) **61** (2014), no. 1, 1–32.

References III

Andrew Chi-Chih Yao, *Separating the polynomial-time hierarchy by oracles*, 26th Annual Symposium on Foundations of Computer Science (sfcs 1985), IEEE, 1985, pp. 1–10.